

# Relativistic viscous hydrodynamics order by order

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## Abstract

In this paper, we propose a method of solving the viscous hydrodynamics order by order in a derivative expansion. In such a method, the zero-order solution is just one of the ideal hydrodynamics. All the other higher order corrections satisfy the same first-order partial differential equations but with different inhomogeneous terms. We take the Bjorken flow as an example to test the validity of our method and present how to deal with the problems about the initial condition and perturbation evolution in our formalism.

PACS numbers: 12.38.Mh, 25.75.-q, 52.27.Ny

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## I. INTRODUCTION

Relativistic hydrodynamics has been an important and useful theoretical tool in high energy heavy-ion physics such as at BNL Relativistic Heavy Ion Collider (RHIC) and CERN Large Hadron Collider (LHC), which have succeeded greatly in describing the collective flow from the data of those colliders [1–8]. Hydrodynamics can be considered as an macroscopic effective field theory of more fundamental microscopic theory such as quantum field theory, in describing the non-equilibrium evolution of a given system. However, it is not trivial to build a consistent and causal relativistic hydrodynamics beyond the ideal hydrodynamics. The relativistic first order viscous hydrodynamics was first proposed by Eckart in [9], and Landau and Lifshitz in [10], both of which showed that dissipative fluctuation may propagate at an infinite speed, which is inconsistent with the relativistics. This is the so-called causality problem. In addition, the solution is also unstable due to small perturbation of the equilibria in these viscous hydrodynamics [11]. The nonrelativistic causal viscous hydrodynamics was first presented by Müller in [12] and was later generalized into the relativistic version by Israel and Stewart in [13]. They remedied the previous viscous hydrodynamics by introducing some second order terms in deviations away from equilibrium into the entropy current. Therefore these formalisms are also referred to as the second order theory of viscous hydrodynamics. For example, the relaxation time for shear viscous tensor, which is one of the well-known second order parameters, describes how long it will take for the system to return to the equilibrium states after a small perturbation via shear viscosity. Therefore, if all these hydrodynamic parameters satisfy certain constraints, the system will be causal and stable (e.g., for shear viscous tensor [14, 17], bulk viscous pressure [15], and heat conducting flow [16, 17]). Recently, some other authors discussed further the second order viscous hydrodynamics [18–20], especially from the point of view of effective theory, in which all the second derivative terms are included.

There exist two different methods solving the viscous hydrodynamic equations, either expressing the dissipative contributions in energy-momentum tensor or charge current in terms of differentials on primary variables such as fluid 4-velocity  $u^\mu$  and chemical potential, then substituting them into the hydrodynamic equation, or regarding the dissipative quantities as independent dynamical quantities which satisfies extra differential equations. The same difficulty in both methods is that the contributions from different orders are mixed

together, which implies small errors in high orders might also cause big uncertainty in the numerical simulations after time evolutions. On the other hand, the point of view of effective theory, where higher order terms should always be small corrections to the lower order during the whole evolution, might give us some hints to simplify these problem. Besides, from the second order to the third or even higher orders, one has to deal with more and more complicated differential hydrodynamic equations.

In Sec. II of this paper, we will try to present a consistent formalism of solving the viscous hydrodynamic equation order by order in comparison with microscopic theories. We will show that the zero order solution is just the one of the ideal hydrodynamics in our method and all the other higher order corrections satisfy the same first-order partial differential equation but with different inhomogeneous source terms. We find that our method is a recursion process, the next order solution can be obtained only after we get all the previous order solution. In every order calculation, we only need to deal with the same first order differential equations with different inhomogeneous source terms. Such method can be manipulated to any higher order. In Sec.III, we will discuss how to deal with the problems about the initial condition and stability in our formalism. In Sec.IV, we choose the Bjorken flow as a test to illustrate the validity of our method and how to manipulate the initial condition and perturbation evolution specifically. Finally, there is the conclusion in Sec V.

## II. HYDRODYNAMICS ORDER BY ORDER

Since we will present our method mainly theoretically or formally, for simplicity, we will restrict ourselves to the conformal non-charged fluid. In such a system, the dissipative terms are constrained greatly due to the conformal symmetry. More general cases can be extended straightforwardly and will be presented elsewhere. Since the fluid is not charged, only energy-momentum conservation is involved,

$$\partial_\nu T^{\nu\mu} = 0, \tag{1}$$

where the energy-momentum tensor  $T^{\mu\nu}$  is assumed to be able to expand as the primary hydrodynamic variable, local fluid velocity  $u^\mu(x)$  ( $u^2 = -1$ ) and local temperature  $T(x)$ . In the following, we will always work in the Landau frame and adopt the convention of the metric tensor  $g^{\mu\nu} = [-1, +1, +1, +1]$ . In such frame and convention, the energy-momentum

tensor can be generally decomposed into

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} + \Pi^{\mu\nu}, \quad (2)$$

where  $\epsilon$  is the energy density,  $P$  is the pressure, and  $\Pi^{\mu\nu}$  includes all the dissipative terms and satisfies  $u_\mu \Pi^{\mu\nu} = 0$ .

Generally, in long wavelength and low frequency limit, if Knudsen number  $K = \ell_{mfp} \partial_\mu \ll 1$ , with  $\ell_{mfp}$  the mean free path of particles and  $\partial_\mu$  the space-time derivatives, the hydrodynamic is workable [19, 20]. In this case, we can expand all hydrodynamic quantities and equations in the power of the Knudsen number. In the leading order, we will get the ideal fluid. In the first order,  $\Pi^{\mu\nu}$  will be introduced and can be expanded as the differentials of the local velocity  $u^\mu$  order by order. In a conformal theory, this dissipative term can be generally written as [18, 21],

$$\Pi^{\mu\nu} = -\eta \sigma^{\mu\nu} + \pi^{\mu\nu}, \quad (3)$$

$$\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha} \Delta^{\nu\beta} \left( \partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} \Delta_{\alpha\beta} \nabla \cdot u \right), \quad (4)$$

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu, \quad \nabla \cdot u \equiv \Delta^{\alpha\beta} \partial_\alpha u_\beta, \quad (5)$$

where  $\eta$  is the shear viscosity,  $\pi^{\mu\nu}$  is the second-order differential terms and in a conformal theory, can be generally decomposed into the following form,

$$\begin{aligned} \pi^{\mu\nu} = & \eta \tau_\Pi \left[ u^\alpha \partial_\alpha \sigma^{\mu\nu} + \frac{1}{3} \sigma^{\mu\nu} \partial_\alpha u^\alpha \right] + \lambda_1 \left[ \sigma^\mu{}_\alpha \sigma^{\nu\alpha} - \frac{1}{3} \Delta^{\mu\nu} \sigma_{\alpha\beta} \sigma^{\alpha\beta} \right] \\ & + \frac{1}{2} \lambda_2 \left[ \sigma^\mu{}_\alpha \Omega^{\nu\alpha} \sigma^\nu{}_\alpha \Omega^{\mu\alpha} \right] + \lambda_3 \left[ \Omega^\mu{}_\alpha \Omega^{\nu\alpha} - \frac{1}{3} \Delta^{\mu\nu} \Omega_{\alpha\beta} \Omega^{\alpha\beta} \right], \end{aligned} \quad (6)$$

$$\Omega_{\mu\nu} \equiv \frac{1}{2} \Delta_{\mu\alpha} \Delta_{\nu\beta} (\partial^\alpha u^\beta - \partial^\beta u^\alpha), \quad (7)$$

where  $\tau_\Pi, \lambda_{1,2,3}$  are transport coefficients in the second order theory and  $\Omega^{\mu\nu}$  is the vorticity tensor. The entropy current  $S^\mu$  is defined as [13]

$$S^\mu = \frac{P}{T} u^\mu - \frac{1}{T} u_\nu T^{\nu\mu} - Q^\mu, \quad (8)$$

where  $Q^\mu$  represents a possible second order correction. In the leading order,  $S^\mu = s u^\mu$ , with  $s$  the entropy density

$$sT = \epsilon + P. \quad (9)$$

Now we will propose our method. Firstly, it is quite natural and straightforward that we will treat fluid 4-velocity  $u^\mu(x)$  and temperature  $T(x)$  as the primary variables, energy density and pressure can be expressed as the function of  $T(x)$  by the equation of state. We can imagine the final solution of  $u^\mu(x)$  and  $T(x)$  can be obtained by the serials expansion as

$$u^\mu(x) = u_0^\mu(x) + u_1^\mu(x) + u_2^\mu(x) + \dots, \quad (10)$$

$$T(x) = T_0(x) + T_1(x) + T_2(x) + \dots, \quad (11)$$

where the series are expanded in the power of Knudsen number. Then, we assume that all hydrodynamic quantities and equations can be expanded in the power of Knudsen number. For example, it follows that the energy-momentum tensor can be expanded as

$$T^{\nu\mu} = T_0^{\nu\mu} + T_1^{\nu\mu} + T_2^{\nu\mu} + \dots \quad (12)$$

The zero-order energy-momentum tensor is given by

$$T_0^{\nu\mu} = (\epsilon_0 + P_0) u_0^\mu u_0^\nu + P_0 g^{\mu\nu} \quad (13)$$

where  $\epsilon_0 \equiv \epsilon(T_0)$  and  $P_0 \equiv P(T_0)$ . It is just the decomposition of the ideal fluid.

Secondly, in order to avoid the mixture of different orders, we assume the differential hydrodynamic equations satisfy the conservation law order by order, i.e. we let

$$\partial_\mu T_i^{\mu\nu} = 0, \quad (i = 0, 1, 2, \dots). \quad (14)$$

It looks very robust and adds more constraints to the hydrodynamic equations, but it is reasonable. From the classical kinetic theory, i.e., the Boltzmann equations, the distribution function  $f$  can be expanded in power of  $K$ ,  $f = f_0 + f_1 + f_2 + \dots$  and obtained order by order. Provided the time reversal symmetry is protected, we can get  $\partial_\mu T_i^{\mu\nu} = 0$  (also see Appendix I). These kinds of methods are widely used in theoretical physics; e.g., for quantum kinetic theory, a similar treatment will give the exact transport coefficients of chiral magnetic and vortical effects [22, 23] or Hall effects [24], and other related hydrodynamics [25–27].)

Back to our case, the zero-order approximation  $u_0^\mu$  and  $T_0$  can be obtained by solving the ideal hydrodynamic equation,

$$\partial_\nu T_0^{\nu\mu} = 0. \quad (15)$$

As usual, we can decompose them into a component parallel to  $u_0^\mu$  by contracting Eq.(15) with  $u_0^\mu$ ,

$$(\epsilon_0 + P_0) \nabla \cdot u_0 + \epsilon'_0 \dot{T}_0 = 0, \quad (16)$$

and the other three components orthogonal to  $u_0^\mu$  by projecting Eq.(15) with  $\Delta_{0\mu\nu}$

$$\Delta_{0\mu\nu} \left( \dot{u}_0^\nu + \frac{1}{T_0} \partial^\nu T_0 \right) = 0 \quad (17)$$

where  $\dot{T}_0 \equiv u_0^\mu \partial_\mu u_0$ ,  $\epsilon'_0 \equiv \frac{d\epsilon}{dT}|_{T=T_0} = \frac{d\epsilon_0}{dT_0}$  and  $\Delta_{0\mu\nu} \equiv g_{\mu\nu} + u_{0\mu} u_{0\nu}$ . The zero-order entropy current is given by

$$S_0^\mu = \frac{P_0}{T_0} u_0^\mu - \frac{1}{T_0} u_{0\nu} T_0^{\nu\mu} = \frac{\epsilon_0 + P_0}{T_0} u_0^\mu = s_0 u_0^\mu. \quad (18)$$

It is well known that it is conserved for the ideal fluid

$$\partial_\mu S_0^\mu = 0. \quad (19)$$

### A. The first-order equations

Now let us continue to deal with the next-to-leading order, in which the energy-momentum tensor is given by

$$T_1^{\nu\mu} = (\epsilon'_0 + P'_0) T_1 u_0^\mu u_0^\nu + P'_0 T_1 g^{\mu\nu} + (\epsilon_0 + P_0) (\bar{u}_1^\mu u_0^\nu + u_0^\mu \bar{u}_1^\nu) - \eta_0 \sigma_0^{\mu\nu} \quad (20)$$

where  $\bar{u}_1^\mu \equiv \Delta_0^{\mu\nu} u_{1\nu}$  and

$$\sigma_0^{\mu\nu} \equiv \Delta^{\mu\alpha} \Delta^{\nu\beta} \left( \partial_\alpha u_{0\beta} + \partial_\beta u_{0\alpha} - \frac{2}{3} \Delta_{\alpha\beta} \nabla \cdot u_0 \right). \quad (21)$$

Note that since there are corrections to the temperature  $T_1(x)$ , the energy density and pressure will also have some corrections,

$$\epsilon(T) = \epsilon_0 + \epsilon'_0 T_1 + O(K^2), \quad P(T) = P_0 + P'_0 T_1 + O(K^2). \quad (22)$$

It should be noted that we have constrained the normalization condition for  $u_1^\mu$  as

$$(u_0 + u_1)^2 = -1, \quad (23)$$

which leads to the following relation:

$$u_0 \cdot u_1 = 1 - \sqrt{1 + \bar{u}_1^2}. \quad (24)$$

Hence only three components of  $u_1^\mu$  are independent and the component parallel to  $u_0^\mu$  can be totally determined by  $\bar{u}_1^\mu$ . In addition, we can notice that  $u_0 \cdot u_1$  only contributes to at least second order, that is why only  $\bar{u}_1^\mu$  is involved in the first-order energy-momentum tensor (20).

Since the zero-order energy-momentum tensor has already satisfied the conservation equation, we need the first-order energy-momentum tensor to satisfy the conservation equation independently

$$\partial_\nu T_1^{\nu\mu} = 0. \quad (25)$$

The component parallel to  $u_0^\mu$  reads

$$\nabla \cdot \bar{u}_1 + \frac{1}{T_0} \left( \frac{1}{v_s^2} + 1 \right) \bar{u}_1^\mu \partial_\mu T_0 + \frac{1}{T_0 v_s^2} \dot{T}_1 + T_1 \left( \frac{1}{T_0 v_s^2} \right)' \dot{T}_0 = \frac{1}{\epsilon_0 + P_0} u_{0\mu} C_1^\mu, \quad (26)$$

where

$$u_{0\mu} C_1^\mu = \eta_0 \sigma_0^{\mu\nu} \partial_\nu u_{0\mu} \quad (27)$$

and the components orthogonal to  $u_0^\mu$  reads

$$\Delta_{0\mu\alpha} (\dot{\bar{u}}_1^\alpha + \bar{u}_1^\nu \partial_\nu u_0^\alpha) + \frac{1}{T_0} \bar{u}_{1\mu} \dot{T}_0 + \frac{1}{T_0} \Delta_{0\mu\alpha} \partial^\alpha T_1 - \frac{T_1}{T_0^2} \Delta_{0\mu\nu} \partial^\nu T_0 = \frac{1}{\epsilon_0 + P_0} \Delta_{0\mu\alpha} C_1^\alpha \quad (28)$$

where

$$\Delta_{0\mu\alpha} C_1^\alpha = \Delta_{0\mu\alpha} \partial_\nu (\eta_0 \sigma_0^{\nu\alpha}). \quad (29)$$

The first-order correction to the entropy current  $S_0^\mu$  is given by

$$S_1^\mu = \frac{1}{T_0} [\epsilon'_0 T_1 u_0^\mu + (\epsilon_0 + P_0) \bar{u}_1^\mu]. \quad (30)$$

It is easy to show that the divergence of the entropy current is always positive and consistent with the second thermal law,

$$\partial_\mu S_1^\mu = \frac{\eta_0}{2T_0} \sigma_0^{\mu\nu} \sigma_{0\mu\nu} \geq 0. \quad (31)$$

## B. The second-order equations

We now turn to the second order, in which the energy-momentum tensor is given by

$$\begin{aligned} T_2^{\nu\mu} = & (\epsilon'_0 + P'_0) T_2 u_0^\mu u_0^\nu + P'_0 T_2 g^{\mu\nu} + (\epsilon_0 + P_0) (\bar{u}_2^\mu u_0^\nu + u_0^\mu \bar{u}_2^\nu) \\ & + \frac{1}{2} T_1^2 [P''_0 g^{\mu\nu} + (\epsilon''_0 + P''_0) u_0^\mu u_0^\nu] + (\epsilon_0 + P_0) (\bar{u}_1^2 u_0^\mu u_0^\nu + \bar{u}_1^\mu \bar{u}_1^\nu) \\ & + (\epsilon'_0 + P'_0) T_1 (\bar{u}_1^\mu u_0^\nu + \bar{u}_0^\mu u_1^\nu) - \eta'_0 T_1 \sigma_0^{\mu\nu} - \eta_0 \sigma_1^{\mu\nu} + \Pi_0^{\nu\mu}, \end{aligned} \quad (32)$$

where  $\bar{u}_2^\mu \equiv \Delta_0^{\mu\nu} u_{2\nu}$ . For energy density and pressure, we have  $\epsilon = \epsilon_0 + \epsilon'_0 T_1 + \epsilon'_0 T_2 + \frac{1}{2} \epsilon''_0 T_1^2$ , and  $P = P_0 + P'_0 T_1 + P'_0 T_2 + \frac{1}{2} P''_0 T_1^2$ . Just as we did for  $u_1^\mu$ , we have constrained the normalization condition for  $u_2^\mu$  as

$$(u_0 + u_1 + u_2)^2 = -1, \quad (33)$$

which results in

$$u_0 \cdot u_2 = \sqrt{1 + \bar{u}_1^2} - \sqrt{1 + \bar{u}_1^2 + 2\bar{u}_1 \cdot \bar{u}_2 + \bar{u}_2^2}. \quad (34)$$

It is easy to show that  $u_0 \cdot u_2$  only contributes to at least third order, which can be dropped off for the second order  $T_2^{\mu\nu}$ . However, we must consider  $u_0 \cdot u_1$  which has been neglected at the first order  $T_1^{\mu\nu}$ . Since both the zero-order and the first order energy-momentum tensors have already satisfy the conservation equation, we need the second-order energy-momentum tensor to satisfy the conservation equation independently, i.e.,

$$\partial_\nu T_2^{\nu\mu} = 0. \quad (35)$$

The component parallel to  $u_0^\mu$  reads,

$$\nabla \cdot \bar{u}_2 + \frac{1}{T_0} \left( \frac{1}{v_s^2} + 1 \right) \bar{u}_2^\mu \partial_\mu T_0 + \frac{1}{T_0 v_s^2} \dot{T}_2 + T_2 \left( \frac{1}{T_0 v_s^2} \right)' \dot{T}_0 = \frac{1}{\epsilon_0 + P_0} u_{0\mu} C_2^\mu, \quad (36)$$

where

$$\begin{aligned} u_{0\mu} C_2^\mu = & \epsilon_0 u_0^\mu \partial_\mu \left[ \frac{(\epsilon_0 + P_0)}{2} \left( \frac{1}{\epsilon_0 + P_0} \right)'' T_1^2 - \bar{u}_1^2 \right] \\ & + u_0^\mu \partial_\mu \left[ \frac{(\epsilon_0 + P_0)}{2} \left( \frac{P_0}{\epsilon_0 + P_0} \right)'' T_1^2 - P_0 \bar{u}_1^2 \right] \\ & - \left[ \epsilon'_0 T_1 u_0^\mu + (\epsilon_0 + P_0) \bar{u}_1^\mu \right] \partial_\mu \left[ \frac{(\epsilon'_0 + P'_0)}{(\epsilon_0 + P_0)} T_1 \right] \\ & - \left[ \left( \frac{\epsilon'_0 + P'_0}{\epsilon_0 + P_0} \eta_0 - \eta'_0 \right) T_1 \sigma_0^{\mu\nu} - \eta_0 \sigma_1^{\mu\nu} + \pi_0^{\nu\mu} \right] \frac{1}{2} \sigma_{0\mu\nu} \\ & + (\epsilon_0 + P_0) \bar{u}_1^\nu u_{0\mu} \partial_\nu \bar{u}_1^\mu, \end{aligned} \quad (37)$$

and the components orthogonal to  $u_0^\mu$  reads

$$\Delta_{0\mu\alpha} (\dot{\bar{u}}_2^\alpha + \bar{u}_2^\nu \partial_\nu u_0^\alpha) + \frac{1}{T_0} \bar{u}_{2\mu} \dot{T}_0 + \frac{1}{T_0} \Delta_{0\mu\alpha} \partial^\alpha T_2 - \frac{T_2}{T_0^2} \Delta_{0\mu\nu} \partial^\nu T_0 = \frac{1}{\epsilon_0 + P_0} \Delta_{0\mu\alpha} C_2^\alpha, \quad (38)$$



where

$$\begin{aligned}
\Delta_{0\mu\alpha} C_2^\alpha = & P_0 \Delta_{0\mu\nu} \partial^\nu \left[ -\frac{(\epsilon_0 + P_0)}{2} \left( \frac{1}{\epsilon_0 + P_0} \right)'' T_1^2 + \bar{u}_1^2 \right] \\
& + \Delta_{0\mu\nu} \partial^\nu \left[ \frac{(\epsilon_0 + P_0)}{2} \left( \frac{P_0}{\epsilon_0 + P_0} \right)'' T_1^2 - P_0 \bar{u}_1^2 \right] \\
& - \Delta_{0\mu\alpha} \partial_\nu \left[ \left( \eta'_0 - \frac{\epsilon'_0 + P'_0}{\epsilon_0 + P_0} \eta_0 \right) \sigma_0^{\nu\alpha} + \eta_0 \sigma_1^{\nu\alpha} - \pi_0^{\nu\alpha} \right] \\
& + T_1^{\nu\alpha} \Delta_{0\mu\alpha} \partial_\nu \left[ \frac{(\epsilon'_0 + P'_0)}{(\epsilon_0 + P_0)} T_1 \right] + \Delta_{0\mu\alpha} \partial_\nu [(\epsilon_0 + P_0) \bar{u}_1^\alpha \bar{u}_1^\nu]. \quad (39)
\end{aligned}$$

The second-order correction to the entropy current is

$$S_2^\mu = -\frac{1}{T_0} u_{0\nu} T_2^{\nu\mu} + \frac{1}{2} \bar{u}_1^2 S_0^\mu - \frac{T_1}{T_0} S_1^\mu - \frac{1}{T_0} \bar{u}_{1\nu} T_1^{\nu\mu} + \frac{T_1}{T_0^2} \left( \frac{1}{2} \epsilon'_0 T_1 u_0^\mu + P'_0 T_0 \bar{u}_1^\mu \right). \quad (40)$$

It is straightforward to derive the rate of entropy production for the second order as

$$\partial_\mu S_2^\mu = \frac{1}{2T_0} (\eta'_0 T_1 \sigma_0^{\mu\nu} + 2\eta_0 \sigma_1^{\mu\nu} - \pi_0^{\mu\nu}) \sigma_{0\mu\nu}. \quad (41)$$

Generally, they are not positive definite, however they do not violate the second law of thermodynamics since the third order terms must be small compared to the second order term in the domain of applicability of hydrodynamics. Similar possible negative signs and comments can also be found in [18].

Actually, such a recursion process can be generalized to any higher orders without any difficulty. It is important to note that all the equations have a similar form, i.e., the component parallel to  $u_0^\mu$  reads

$$\nabla \cdot \bar{u}_n + \frac{1}{T_0} \left( \frac{1}{v_s^2} + 1 \right) \bar{u}_n^\mu \partial_\mu T_0 + \frac{1}{T_0 v_s^2} \dot{T}_n + T_n \left( \frac{1}{T_0 v_s^2} \right)' \dot{T}_0 = \frac{1}{\epsilon_0 + P_0} u_{0\mu} C_n^\mu, \quad (42)$$

and the components orthogonal to  $u_0^\mu$  reads

$$\Delta_{0\mu\alpha} (\dot{u}_n^\alpha + \bar{u}_n^\nu \partial_\nu u_0^\alpha) + \frac{1}{T_0} \bar{u}_{n\mu} \dot{T}_0 + \frac{1}{T_0} \Delta_{0\mu\alpha} \partial^\alpha T_n - \frac{T_n}{T_0^2} \Delta_{0\mu\nu} \partial^\nu T_0 = \frac{1}{\epsilon_0 + P_0} \Delta_{0\mu\alpha} C_n^\alpha, \quad (43)$$

where  $C_n^\alpha$  depends only on the  $u_0^\mu$ ,  $T_0$ ,  $\bar{u}_m^\mu$ , and  $T_m$   $1 \leq m \leq n-1$  or their derivatives. It should be pointed out that our process is very similar to the method used in [25–27].

### III. INITIAL CONDITIONS AND STABILITY

In order to solve the hydrodynamic equations, we must give some specific initial conditions, e.g.,  $u(t_0, \vec{x})$  and  $T(t_0, \vec{x})$ , where  $t_0$  is the initial time. Generally, we can decompose

them into

$$\begin{aligned} u^\mu(t_0, \vec{x}) &= u_0^\mu(t_0, \vec{x}) + u_1^\mu(t_0, \vec{x}) + u_2^\mu(t_0, \vec{x}) + \dots, \\ T(t_0, \vec{x}) &= T_0(t_0, \vec{x}) + T_1(t_0, \vec{x}) + T_2(t_0, \vec{x}) + \dots, \end{aligned} \quad (44)$$

in any way as long as they satisfy

$$u_0^\mu(t_0, \vec{x}) \gg u_1^\mu(t_0, \vec{x}) \gg u_2^\mu(t_0, \vec{x}) \gg \dots, \quad (45)$$

$$T_0(t_0, \vec{x}) \gg T_1(t_0, \vec{x}) \gg T_2(t_0, \vec{x}) \gg \dots, \quad (46)$$

With different decompositions, the final result should differ only in higher orders. For simplicity, we can just set

$$\begin{aligned} u_0^\mu(t_0, \vec{x}) &= u^\mu(t_0, \vec{x}), \quad u_1^\mu(t_0, \vec{x}) = 0, \quad u_2^\mu(t_0, \vec{x}) = 0, \quad \dots \\ T_0(t_0, \vec{x}) &= T(t_0, \vec{x}), \quad T_1(t_0, \vec{x}) = 0, \quad T_2(t_0, \vec{x}) = 0, \quad \dots \end{aligned} \quad (47)$$

With the initial state  $u_0^\mu(t_0, \vec{x})$ , we can solve the zeroth-order equations (16) and (17) and obtain the solution  $u_0^\mu(t, \vec{x})$ . With this zeroth-order solution, we can calculate the first-order inhomogeneous term  $u_{0\mu}C_1'^\mu$  and  $\Delta_{0\mu\alpha}C_1^\alpha$  which includes the first derivative of  $u_0^\mu(t, \vec{x})$  and solve the first-order equations (42) and (43) under the initial conditions  $u_1^\mu(t_0, \vec{x}) = 0$  and  $T_1(t_0, \vec{x}) = 0$ . After getting the first-order solution, we can proceed further to obtain the second-order contribution and so on. Hence, to solve the  $n$ th-order equations, there is no need to know the initial value of the derivative of  $n$ th-order correction; we only need the derivative of lower order corrections which have been solved already. This should be a good advantage in our iterative method compared to other methods. Besides, using the initial conditions (47), we actually rule out all the free modes which will lead to instability from the homogenous solutions in Eqs.(42) and (43) for  $n \geq 1$ . Only the particular solution which is proportional to the inhomogeneous term survives. However, in the numerical simulation, the computation error can be inevitable and make the above argument invalid. The interesting thing in our method is that whether the instability arises or not depends only on the zeroth-order solution  $u_0^\mu$ , as shown in Eq.(42) and Eq.(43). In the next section, we will use Bjorken flow as a simple example to illustrate how the perturbations evolve.

#### IV. BJORKEN FLOW

In this section, we will choose the (1+1)-dimensional Bjorken flow [28] as an example to illustrate the validity of our formalism. In order to do that, we will use the coordinate system, such that

$$\tau = \sqrt{t^2 - z^2}, \quad \eta = \tanh^{-1} \frac{z}{t} = \frac{1}{2} \ln \frac{t+z}{t-z}. \quad (48)$$

Bjorken flow is given by

$$u^\mu(\tau) = (\cosh \eta, 0, 0, \sinh \eta). \quad (49)$$

In the following, we will explicitly solve for the velocity field  $u^\mu$  and the energy density  $\epsilon$  or temperature  $T$  with the initial conditions

$$u^\mu(\tau_0) = (\cosh \eta, 0, 0, \sinh \eta), \quad \epsilon(\tau_0) = 3P(\tau_0) = aT^4(\tau_0) = \frac{C}{\tau_0^{4/3}}, \quad (50)$$

where  $C$  is a constant and we have used  $\epsilon_0 = 3P_0 = aT_0^4$  with  $a$  a constant for the conformal fluid. We will choose the decomposition in Eq.(47), i.e.,

$$\begin{aligned} u_0^\mu(\tau_0) &= u^\mu(\tau_0), & u_1^\mu(\tau_0) &= 0, & u_2^\mu(\tau_0) &= 0, & \dots, \\ T_0(\tau_0) &= T(\tau_0), & T_1(\tau_0) &= 0, & T_2(\tau_0) &= 0, & \dots \end{aligned} \quad (51)$$

Hence we have designated full initial configuration to the the zeroth-order equations or ideal hydrodynamic equations (16) and (17). From the uniqueness of the solution for the differential equations, the solution must be the Bjorken's solution

$$u^\mu(\tau) = (\cosh \eta, 0, 0, \sinh \eta), \quad \epsilon_0(\tau) = 3P_0(\tau) = aT_0^4(\tau) = \frac{C}{\tau^{4/3}}. \quad (52)$$

Substituting the Bjorken solutions (52) into the first-order equations (42) and (43) yields

$$\nabla \cdot \bar{u}_1 + \frac{3}{T_0} \partial_\tau T_1 + \frac{1}{\tau T_0} T_1 = \frac{\eta_0}{\epsilon_0 \tau^2}, \quad (53)$$

$$\partial_\tau \bar{u}_{1\mu} + \frac{v_{0\mu}}{\tau} (v_0 \cdot \bar{u}_1) - \frac{1}{3\tau} \bar{u}_{1\mu} + \frac{1}{T_0} \Delta_{0\mu\alpha} \partial^\alpha T_1 = 0, \quad (54)$$

where  $v_0^\mu = (\frac{z}{\tau}, 0, 0, \frac{t}{\tau})$ . Given the initial condition  $T_1(\tau_0) = 0$  and  $\bar{u}_1^\mu(\tau_0) = (0, 0, 0, 0)$ , we can solve the equations and obtain

$$\bar{u}_1^\mu = (0, 0, 0, 0), \quad T_1 = \frac{\hat{\eta}_0}{2\tau_0^{2/3}} \left[ 1 - \left( \frac{\tau_0}{\tau} \right)^{2/3} \right] T_0, \quad (55)$$

where  $\hat{\eta}_0$  and  $\kappa$  are both constants and defined as in Ref.[18]

$$\eta_0 = bT_0^3 = C\hat{\eta}_0 \left(\frac{\epsilon_0}{C}\right)^{3/4} = \frac{C\hat{\eta}_0}{\tau}, \quad \kappa = \left(\frac{C}{a}\right)^{1/4}. \quad (56)$$

The energy density of the first order can be given by

$$\epsilon_1 = \frac{2\hat{\eta}_0}{\tau_0^{2/3}} \left[1 - \left(\frac{\tau_0}{\tau}\right)^{2/3}\right] \epsilon_0. \quad (57)$$

Now substituting the first-order solution (55) into the second-order equations (36) and (38), we can have

$$\nabla \cdot \bar{u}_2 + \frac{3}{T_0} \partial_\tau T_2 + \frac{1}{\tau T_0} T_2 = \frac{2(\hat{\eta}_0 \hat{\tau}_{\Pi 0} - \hat{\lambda}_{1,0})}{3\tau^{7/3}}, \quad (58)$$

$$\partial_\tau \bar{u}_{2\mu} + \frac{v_{0\mu}}{\tau} (v_0 \cdot \bar{u}_2) - \frac{1}{3\tau} \bar{u}_{2\mu} + \frac{1}{T_0} \Delta_{0\mu\alpha} \partial^\alpha T_2 = 0, \quad (59)$$

where

$$\tau_{\Pi 0} = \hat{\tau}_{\Pi 0} \left(\frac{\epsilon_0}{C}\right)^{-1/4}, \quad \lambda_{1,0} = C\hat{\lambda}_{1,0} \left(\frac{\epsilon_0}{C}\right)^{1/2}. \quad (60)$$

Again with the initial condition  $T_2(\tau_0) = 0$  and  $\bar{u}_2^\mu(\tau_0) = (0, 0, 0, 0)$ , the solution is given by

$$\bar{u}_2^\mu = (0, 0, 0, 0), \quad T_2 = \frac{(\hat{\eta}_0 \hat{\tau}_{\Pi 0} - \hat{\lambda}_{1,0})}{6\tau_0^{4/3}} \left[1 - \left(\frac{\tau_0}{\tau}\right)^{4/3}\right] T_0, \quad (61)$$

or the energy density

$$\epsilon_2 = \frac{3\hat{\eta}_0^2}{2\tau_0^{4/3}} \left[1 - \left(\frac{\tau_0}{\tau}\right)^{2/3}\right]^2 \epsilon_0 + \frac{2(\hat{\eta}_0 \hat{\tau}_{\Pi 0} - \hat{\lambda}_{1,0})}{3\tau_0^{4/3}} \left[1 - \left(\frac{\tau_0}{\tau}\right)^{4/3}\right] \epsilon_0. \quad (62)$$

Up to the second-order contribution, the energy density is given by

$$\begin{aligned} \epsilon = \frac{C}{\tau^{4/3}} & \left\{ 1 + \frac{\hat{\eta}_0}{2\tau_0^{2/3}} \left[1 - \left(\frac{\tau_0}{\tau}\right)^{2/3}\right] + \frac{3\hat{\eta}_0^2}{2\tau_0^{4/3}} \left[1 - \left(\frac{\tau_0}{\tau}\right)^{2/3}\right]^2 \right. \\ & \left. + \frac{2(\hat{\eta}_0 \hat{\tau}_{\Pi 0} - \hat{\lambda}_{1,0})}{3\tau_0^{4/3}} \left[1 - \left(\frac{\tau_0}{\tau}\right)^{4/3}\right] \right\}. \end{aligned} \quad (63)$$

It is obvious that our expansion method will be valid as long as

$$\frac{\hat{\eta}_0}{\tau_0^{2/3}} \ll 1, \quad \frac{|\hat{\eta}_0 \hat{\tau}_{\Pi 0} - \hat{\lambda}_{1,0}|}{\tau_0^{4/3}} \ll 1. \quad (64)$$

Our result in Eq.(63) is consistent with the result obtained in Ref.[18] once we drop the terms including  $\tau_0$  which enters due to the constraint of the initial conditions.

Now let us take into account the stability problems in this specific example. We will follow the method given by Gubser and Yarom in Ref.[29]. In order to do that, we rewrite the same homogeneous differential equations corresponding to Eqs.(53) and (54) or Eqs.(58) and (59) or even higher orders as

$$\partial_\tau \delta \hat{T} + \frac{1}{3\tau} \partial_\eta \delta u_\eta + \frac{1}{3} \nabla_\perp \cdot \delta \mathbf{u}_\perp = 0, \quad (65)$$

$$\partial_\tau \delta u_\eta + \frac{2}{3\tau} \delta u_\eta + \frac{1}{\tau} \partial_\eta \delta \hat{T} = 0, \quad (66)$$

$$\partial_\tau \delta \mathbf{u}_\perp - \frac{1}{3\tau} \delta \mathbf{u}_\perp + \nabla_\perp \delta \hat{T} = 0, \quad (67)$$

where  $\delta \hat{T} = \delta T/T_0$ ,  $\delta u_\eta = v_0 \cdot \delta u$  and  $\delta \mathbf{u}_\perp = (0, \delta u_x, \delta u_y, 0)$ . Here we have used  $\delta T$  and  $\delta u_\mu$  to denote  $T_1, T_2, \dots, T_n$  and  $\bar{u}_{1\mu}, \bar{u}_{2\mu}, \dots, \bar{u}_{n\mu}$  respectively. In the momentum space,

$$\delta \hat{T} = \int \delta \hat{\mathcal{T}} e^{ik_\eta \eta + i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} dk_\eta d^2 \mathbf{k}_\perp, \quad \delta \hat{u}_\mu = \int \delta \hat{\mathcal{U}}_\mu e^{ik_\eta \eta + i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} dk_\eta d^2 \mathbf{k}_\perp. \quad (68)$$

It follows that

$$\partial_\tau \delta \hat{\mathcal{T}} + \frac{i}{3\tau} k_\eta \delta U_\eta + \frac{i}{3} \mathbf{k}_\perp \cdot \delta \mathbf{U}_\perp = 0, \quad (69)$$

$$\partial_\tau \delta U_\eta + \frac{2}{3\tau} \delta U_\eta + \frac{i}{\tau} k_\eta \delta \hat{\mathcal{T}} = 0, \quad (70)$$

$$\partial_\tau \delta \mathbf{U}_{\perp\mu} - \frac{1}{3\tau} \delta \mathbf{U}_{\perp\mu} + i\mathbf{k}_{\perp\mu} \delta \hat{\mathcal{T}} = 0. \quad (71)$$

We can decompose  $\delta \mathbf{U}_{\perp\mu}$  into

$$\delta \mathbf{U}_\perp = \mathbf{k}_\perp \delta W + \tilde{\mathbf{k}}_\perp \delta \tilde{W}, \quad (72)$$

where  $\tilde{\mathbf{k}}_\perp$  is a constant transverse vector satisfying  $\mathbf{k}_\perp \cdot \tilde{\mathbf{k}}_\perp = 0$ . Then we find  $\delta \tilde{W}$  decouples with the other functions

$$\partial_\tau \delta \hat{\mathcal{T}} + \frac{i}{3\tau} k_\eta \delta U_\eta + \frac{i}{3} k_\perp^2 \delta W = 0, \quad (73)$$

$$\partial_\tau \delta U_\eta + \frac{2}{3\tau} \delta U_\eta + \frac{i}{\tau} k_\eta \delta \hat{\mathcal{T}} = 0, \quad (74)$$

$$\partial_\tau \delta W + i\delta \hat{\mathcal{T}} - \frac{1}{3\tau} \delta W = 0, \quad (75)$$

$$\partial_\tau \delta \tilde{W} - \frac{1}{3\tau} \delta \tilde{W} = 0. \quad (76)$$

The solution for  $\delta\tilde{W}$  is given by

$$\delta\tilde{W} = \delta\tilde{W}_0 \left( \frac{\tau}{\tau_0} \right)^{1/3}. \quad (77)$$

We cannot get the analytic solutions for the other functions with the arbitrary  $k_\eta$  and  $k_\perp$ . However, we can take two limits  $k_\eta = 0$  and  $k_\perp = 0$ . When  $k_\eta = 0$ , we can have

$$\delta W = C_1 \tau^{1/3} J_{\frac{2}{3}} \left( \frac{k_\perp \tau}{3} \right) + C_2 \tau^{1/3} N_{\frac{2}{3}} \left( \frac{k_\perp \tau}{3} \right), \quad (78)$$

$$\delta U_\eta = C_3 \frac{1}{\tau^{2/3}}, \quad \delta \hat{\mathcal{T}} = i \left( \partial_\tau - \frac{1}{3\tau} \right) \delta W, \quad (79)$$

where  $J_{\frac{2}{3}}$  and  $N_{\frac{2}{3}}$  denote Bessel and Neumann functions respectively and  $C_1, C_2$ , and  $C_3$  are all integration constants. When  $k_\perp = 0$ , we can have the solution

$$\delta U_\eta = C_4 \left( \frac{1}{\tau} \right)^{\frac{1+\sqrt{1-3k_\eta^2}}{3}} + C_5 \left( \frac{1}{\tau} \right)^{\frac{1-\sqrt{1-3k_\eta^2}}{3}}, \quad (80)$$

$$\delta \hat{\mathcal{T}} = \frac{i\tau}{k_\eta} \left( \partial_\tau + \frac{2}{3\tau} \right) \delta U_\eta, \quad (81)$$

$$\delta W = C_6 \tau^{1/3} + \frac{iC_4}{k_\eta} \tau^{\frac{2-\sqrt{1-3k_\eta^2}}{3}} + \frac{iC_5}{k_\eta} \tau^{\frac{2+\sqrt{1-3k_\eta^2}}{3}}, \quad (82)$$

where  $C_4, C_5$ , and  $C_6$  are also integration constants. From the results of both limits, we can noticed that the perturbations  $\delta U_\eta$  and  $\delta \hat{\mathcal{T}}$  always decay with the proper time  $\tau$  increasing. The perturbation  $\delta\tilde{W}$  increases as  $\tau^{1/3}$  with the proper time. The evolution of the perturbation  $\delta W$  is more complicated and depends on the specific  $k_\eta$  and  $k_\perp$ . However, it is obvious that there is no exponential increase and the behavior of the perturbation increase must be less than the first order of  $\tau$ .

## V. CONCLUSION

In this paper, we have presented a perturbative procedure for solving the viscous hydrodynamic equation order by order in the framework of an effective theory.

For simplicity, we only consider a conformal fluid and more general cases can be straightforward to be obtained. Firstly, we expand all hydrodynamic quantities and differential equations in the power of the Knudsen number. Secondly, we assume the conservation equations are satisfied order by order independently. In the leading order, we get the solutions of an ideal fluid. By solving the differential equations at first and second order, we find these

equations have a uniform expression with different sources. Therefore, we argued that our method can be extended to any orders. We have taken the Bjorken flow as an example and found that our method is very powerful and has good advantage to deal with the initial condition and perturbation evolution. It should be noticed that in our current work we limited ourselves to the theoretical analysis; we postpone the complete numerical analysis and manipulation to a future study.

### Acknowledgments

J.H.G. was supported in part by the Major State Basic Research Development Program in China (Grant No. 2014CB845406), the National Natural Science Foundation of China under the Grant No. 11105137, 11475104 and CCNU-QLPL Innovation Fund (QLPL2014P01). S.P was supported in part by the NSFC under the Grant No. 11205150.

## I. ORDER EXPANSION IN KINETIC THEORY

Our method is inspired by microscopic kinetic theory. As a macroscopic effective theory, hydrodynamic equations can be obtained from other microscopic theories. In most of those microscopic theories, the differential equations are expanded in terms of scaling, then are solved in each order independently. As an example, let us consider the relativistic kinetic theory Boltzmann equations without external fields,

$$\frac{df}{dt} \equiv \frac{p^\mu}{E_p} \partial_\mu f = \mathcal{C}[f], \quad (1)$$

where  $f$  is the distribution functions of particles,  $p^\mu = (E_p, \mathbf{p})$  is the four-momentum of particles and  $\mathcal{C}[f]$  is the collision term. We can expand  $f$  and  $\mathcal{C}[f]$  in a gradient expansion way,  $f = f_0 + f_1 + \dots$ ,  $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$ , which is equivalent to expanding in powers of  $K$ . For simplicity, we neglect the higher order terms in the collision term, and simply set  $\mathcal{C} = \mathcal{C}_0$ . In this case, the current and energy-momentum tensor in each order are given by the integration over momentum, i.e.,  $j_n^\mu = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{E_p} f_n$  and  $T_n^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p} f_n$ , where the lower index  $n$  means the  $n$ -th order in the gradient expansion. Taking covariant derivatives, we get,

$$\begin{aligned} \partial_\mu j_n^\mu &= \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{E_p} \partial_\mu f_n = \int \frac{d^3p}{(2\pi)^3} \mathcal{C}[f_{n-1}] = 0, \\ \partial_\mu T_n^{\mu\nu} &= \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p} \partial_\mu f_n = \int \frac{d^3p}{(2\pi)^3} p^\nu \mathcal{C}[f_{n-1}] = 0, \end{aligned} \quad (2)$$

where in the last line, we used the results of the time-reversal symmetry of the collision term, which guarantees total energy-momentum and number conservation. That implies that in each order the currents and energy-momentum tensor are conserved independently, which is very similar to our method.

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